

ON THE COVARIANCE STRUCTURE OF THE MAXIMUM OF A BROWNIAN MOTION WITH VARIABLE DRIFT

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ABSTRACT. Let B denote a Brownian motion and let f be a continuous function. The purpose of this short note is to prove that the covariance between the maximum of $B + f$ on $[0, t]$ and $B(t)$ equals the expected value of the location of the maximum.

1. INTRODUCTION

Theorem 1. *Let $(B(s), s \geq 0)$ be a Brownian motion with diffusion coefficient $\sigma > 0$, and let $(f(s), s \geq 0)$ be a deterministic continuous function. Define*

$$M(t) := \max_{s \in [0, t]} \{B(s) + f(s)\} \quad \text{and} \quad Z(t) := \max \{s \in [0, t] : B(s) + f(s) = M(t)\} .$$

Then

$$\text{Cov}(M(t), B(t)) = \sigma^2 \mathbb{E}Z(t) , \tag{1}$$

for all $t \geq 0$.

The surprising aspect of (1) is that it gives the same result as if Z were independent of B (notice that $M = B(Z) + f(Z)$, but Z is not independent of B). The proof, which is performed in Section 2, does not give any path interpretation of (1). It simply consists of computing the expected value of the directional derivative of M , with respect to the identity function, in two different ways. It would be interesting to understand it from a different perspective, which could give us some more insight into the joint distribution of B and M . For instance, in the classical case where $f \equiv 0$, one could wonder how to understand (1) in the light of Levy's $M - B$ theorem, or Pitman's $2M - B$ theorem.

In Section 3 we will discuss some related results and also describe the connection between (1) and the integration by parts formula in Malliavin calculus. Although, the motivation and the technique used by the author come from [2], where an analog identity was proved in the Hammersley last-passage percolation model. This identity has also appeared in some other related models [1, 3, 5], and it is a key tool for proving (via hydrodynamical methods) the cube-root asymptotics along the so called characteristic direction of the system.

2. PROOF OF THEOREM 1

For a continuous function h denote

$$M = M(h, t) = \max_{s \in [0, t]} h(s),$$

and

$$Z = Z(h, t) = \max \{s \in [0, t] : h(s) = M\}.$$

Lemma 1 shows that the directional derivative of the functional M , with respect to the identity function, is given by the right-most point in the $\arg \max$ of h .

Lemma 1. For $\epsilon > 0$ let

$$M^\epsilon = M^\epsilon(h, t) = \max_{s \in [0, t]} \{h(s) + \epsilon s\},$$

and

$$Z^\epsilon = Z^\epsilon(h, t) = \max \{s \in [0, t] : h(s) + \epsilon s = M^\epsilon\}.$$

Then

$$\lim_{\epsilon \rightarrow 0^+} Z^\epsilon = Z, \quad (2)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \frac{M^\epsilon - M}{\epsilon} = Z. \quad (3)$$

Proof. Since h is continuous,

$$\begin{aligned} M^\epsilon &= h(Z^\epsilon) + \epsilon Z^\epsilon \\ &= h(Z) + \epsilon Z + \epsilon(Z^\epsilon - Z) - (h(Z) - h(Z^\epsilon)) \\ &\leq M^\epsilon + \epsilon(Z^\epsilon - Z) - (h(Z) - h(Z^\epsilon)), \end{aligned}$$

which show that

$$0 \leq \epsilon(Z^\epsilon - Z) - (h(Z) - h(Z^\epsilon)). \quad (4)$$

We also have that

$$Z^\epsilon \geq Z, \quad (5)$$

since, otherwise,

$$M^\epsilon = h(Z^\epsilon) + \epsilon Z^\epsilon < h(Z^\epsilon) + \epsilon Z \leq h(Z) + \epsilon Z \leq M^\epsilon,$$

that is a contradiction.

Now, by (5), if (2) is not true, then there exist $\delta > 0$ and a sequence $\epsilon_n \rightarrow 0^+$ such that

$$Z^{\epsilon_n} \geq Z + \delta,$$

for all $n \geq 1$. By compactness of $[0, t]$, one can find a subsequence $\epsilon_{n_k} \rightarrow 0^+$ and $\tilde{Z} \in [0, t]$ such that

$$\tilde{Z} = \lim_{k \rightarrow \infty} Z^{\epsilon_{n_k}} \geq Z + \delta > Z.$$

By (4), this implies that

$$h(\tilde{Z}) \geq h(Z),$$

which leads to a contradiction, since Z is the right most point that maximizes h , and the proof of (2) is completed.

To prove (3), notice that

$$M + \epsilon Z \leq M^\epsilon \leq h(Z) + \epsilon Z^\epsilon = M + \epsilon Z + \epsilon(Z^\epsilon - Z),$$

and hence,

$$0 \leq \frac{M^\epsilon - M}{\epsilon} - Z \leq Z^\epsilon - Z \leq t. \quad (6)$$

Together with (2), (6) implies (3).

□

In the next lemma, we take $h = B + f$ and compute the derivative of the expected value of M^ϵ . This computation could be done using Girsanov's theorem (see Section 3). For sake of simplicity, we will present a proof that only requires basic knowledge of Brownian motion, and that holds for any functional $M = M(B)$ of Brownian motion.

Lemma 2. *Let $M = M(B)$ be a random variable on the Wiener space. Define*

$$m(\epsilon) := \mathbb{E}M^\epsilon,$$

where

$$M^\epsilon := M(B^\epsilon) \quad \text{and} \quad B^\epsilon(s) = B(s) + \epsilon s, \quad \text{for } s \in [0, t].$$

Then

$$m'(0) = \frac{1}{\sigma^2} \text{Cov}(M, B(t)). \quad (7)$$

Proof. Let

$$B^\epsilon(s) = B(s) + \epsilon s.$$

Notice that

$$B(s) \stackrel{\text{dist.}}{=} Ns + B_0(s), \quad \text{for } s \in [0, t],$$

where B_0 is a standard Brownian bridge with $B_0(0) = B_0(t) = 0$, and N is a independent Normal random variable of mean 0 and variance σ^2 . Thus,

$$B^\epsilon(s) \stackrel{\text{dist.}}{=} (\epsilon + N)s + B_0(s), \quad \text{for } s \in [0, t].$$

Since $B^\epsilon(t) = u$ if, and only if, $N + \epsilon = u/t$, we have that

$$B^\epsilon(s) \stackrel{\text{dist.}}{=} \frac{u}{t}s + B_0(s), \quad \text{for } s \in [0, t], \quad (8)$$

conditioned on the event that $B^\epsilon(t) = u$. Now, let

$$g(\epsilon, u) = \frac{1}{\sigma\sqrt{2\pi t}} \exp \left\{ -\frac{(u - \epsilon t)^2}{2t\sigma^2} \right\}.$$

By (8), the conditional expectation of M^ϵ , given that $B^\epsilon(t) = u$, does not depend on $\epsilon > 0$. Hence

$$\begin{aligned} m'(\epsilon) &= \int \mathbb{E}(M^\epsilon \mid B^\epsilon(t) = u) \frac{\partial g}{\partial \epsilon}(\epsilon, u) du \\ &= \int \mathbb{E}(M^\epsilon \mid B^\epsilon(t) = u) \left\{ \frac{1}{\sigma^2} (u - \epsilon t) \right\} g(\epsilon, u) du \\ &= \frac{1}{\sigma^2} \left\{ \mathbb{E}(M^\epsilon B^\epsilon(t)) - \epsilon t \mathbb{E}(M^\epsilon) \right\}, \end{aligned}$$

which proves (7). □

By (3), (6), and the Dominated Convergence Theorem,

$$m'(0) = \mathbb{E}Z.$$

Together with Lemma 2, this proves Theorem 1.

3. FINAL REMARKS

Two-sided Brownian motion. If one considers an analog optimization problem with respect to a two-sided Brownian motion on $[-t, t]$, then

$$\text{Cov}(M(t), B(t)) = \sigma^2 \mathbb{E}Z_+(t). \quad (9)$$

(Z_+ stands for the positive part of Z .) The proof of (9) is very similar. Define

$$h^{\epsilon,+}(s) = \begin{cases} h(s) + \epsilon s & \text{if } s \geq 0, \\ h(s) & \text{if } s < 0, \end{cases}$$

and

$$M^{\epsilon,+}(t) = \max_{s \in [-t, t]} \{h^{\epsilon,+}(s)\}.$$

Then

$$\lim_{\epsilon \rightarrow 0^+} \frac{M^{\epsilon,+} - M}{\epsilon} = Z_+.$$

Noticing that the dependence on ϵ only comes from the positive axis, we also have that, for $m(\epsilon) = \mathbb{E}M^{\epsilon,+}$,

$$m'(0) = \frac{1}{\sigma^2} \text{Cov}(M(t), B(t)),$$

which implies (9).

Summing an independent process. A similar formula holds when f is a continuous process that is independent of the Brownian motion.

Negative drift. If we have a strong negative drift and the maximum is attained in a neighborhood of 0 (e.g. Brownian motion minus a parabola [4]) then

$$\lim_{t \rightarrow \infty} \text{Cov}(M, B(t)) = \sigma^2 \mathbb{E}Z. \quad (10)$$

(M denotes the maximum over the positive axis and Z its arg max.)

Malliavin Calculus. Let $\Omega = C[0, t]$ be the space of continuous functions on $[0, t]$ equipped with the Wiener measure \mathbb{P} , and let $F : \Omega \rightarrow \mathbb{R}$ be a random variable. Given a square integrable function ϕ let

$$\psi(t) := \int_0^t \phi(s) ds.$$

The integration by parts formula in Malliavin Calculus states that

$$\mathbb{E}(\langle DF, \psi \rangle) = \mathbb{E} \left(F \int_0^t \phi(s) dB(s) \right). \quad (11)$$

Here, the left-hand side of (11) is the Malliavin derivative of the random variable F in the direction ψ ,

$$\mathbb{E}(\langle DF, \psi \rangle) := \lim_{\epsilon \rightarrow 0^+} \frac{\mathbb{E}F(B + \epsilon\psi) - \mathbb{E}F(B)}{\epsilon},$$

and the integral appearing on the right hand side should be interpreted as an Itô integral. This formula is a consequence of Girsanov's theorem. If ψ is a strictly increasing function then the same reasoning to prove (3) shows that

$$\lim_{\epsilon \rightarrow 0^+} \frac{M(B + f + \epsilon\psi) - M(B + f)}{\epsilon} = \psi(Z).$$

In view of (11), this implies that

$$\mathbb{E}(\psi(Z)) = \mathbb{E} \left(M \int_0^t \phi(s) dB(s) \right). \quad (12)$$

Chain rule. Another possible generalization of (1) is to consider an increasing function H and then use the chain rule (and Lemma 1) to obtain

$$\mathbb{E}(H(M)B) = \sigma^2 \mathbb{E}(H'(M)Z). \quad (13)$$

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REFERENCES

- [1] M. Balázs, E. Cator and T. Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.* **11**, 1094-1132 (2006).
- [2] E. Cator and P. Groeneboom. Second class particles and cube root asymptotics for Hammersleys process. *Ann. Probab.* **34**, 1273-1295 (2006).
- [3] P. A. Ferrari and L. R. G. Fontes. Current fluctuations for the asymmetric simple exclusion process. *Ann. Probab.* **22**, 820-832 (1994).
- [4] P. Groeneboom. Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Realt. Fields* **89**, 79-109 (1985).
- [5] T. Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.*, **40** 19-73 (2012).

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